Symmetric groups (D+F 1.3)

In the last section, we showed how you can think of D_{2n} as a set of some of the bijections from $\{1, 2, \ldots, 2n\}$ to itself. Of course, we need to preserve the geometric structure, so we don't include all bijections.

Let Ω be a set. Let S_Ω be the set of bijections from Ω to itself. Recall, S_Ω is a group w/ operation composition. It's called the <u>symmetric group on Ω</u>.

When $\Omega = \{1, \ldots, n\}$, we denote S_Ω by just S_n.

Note: we can think of S_n as the permutations of $\{1, \ldots, n\}$, so $|S_n| = n!$

Cycle decomposition

Instead of describing elements of S_n by listing where it sends $1, \ldots, n$, we have more efficient notation.

A <u>cycle</u> is a string of integers, written $(a_1 a_2 \ldots a_m)$, which represents the element of S_n that sends a_i to a_{i+1} (for $i = 1, \ldots, m-1$), a_m to a_1, and fixes the rest of $1, \ldots, n$.

Ex: $(1\ 3\ 2) \in S_4$ represents the permutation

$$\begin{aligned} 1 &\mapsto 3 \\ 3 &\mapsto 2 \\ 2 &\mapsto 1 \\ 4 &\mapsto 4 \end{aligned}$$

We can write an arbitrary element of S_n as a product of k cycles $(\underbrace{a_1 \ \dots \ a_{m_1}}_{\text{"length" } m_1})(a_{m_1+1} \ \dots \ a_{m_2})\dots(a_{m_{k-1}+1} \ \dots \ a_{m_k})$, called the <u>cycle decomposition</u>.

Ex: $(1\ 2)(3\ 4) \in S_4$ represents

$$\begin{aligned} 1 &\mapsto 2 \\ 2 &\mapsto 1 \\ 3 &\mapsto 4 \\ 4 &\mapsto 3 \end{aligned}$$

In order to keep notation consistent, we need an algorithm to write each elt of S_n as the product of disjoint cycles.

To demonstrate how this works, define $\sigma \in S_8$ by:

$\sigma(1) = 5$	$\sigma(5) = 4$
$\sigma(2) = 2$	$\sigma(6) = 3$
$\sigma(3) = 6$	$\sigma(7) = 8$
$\sigma(4) = 1$	$\sigma(8) = 7$

Cycle decomposition algorithm for elements of S_n

1.) Pick the smallest, a, of $\{1, 2, \dots, n\}$ which has not already appeared in a previous cycle. Begin the new cycle w/ a. If all $\{1, \dots, n\}$ have appeared, goto step 4.

Ex: $(1$

2.) If $\sigma(a) = a$, close the cycle and go to step 1. Otherwise the cycle continues w/ $b := \sigma(a)$

Ex: $(1\ 5$

3.) If $\sigma(b) = a$, close the cycle to complete it and go to step 1. Otherwise, continue the cycle w/ $c := \sigma(b)$. Repeat this step w/ c as the new value for b until the cycle closes.

Ex: $(1\ 5\ 4)(2)(3\ 6)(7\ 8)$

4.) Remove all cycles of length 1.

Ex: $(1\ 5\ 4)(3\ 6)(7\ 8)$

Multiplying elements of S_n

Since we treat elements of S_n as functions, we multiply from right to left, so if $\sigma = (2\ 5\ 3)(1\ 4)$, $\tau = (1\ 2\ 4)$ Then $\sigma\tau = (1\ 5\ 3\ 2)$ and $\tau\sigma = (2\ 5\ 3\ 4)$.

Claim: Disjoint cycles in S_n commute.

Pf: If σ and τ are disjoint, and $i \in \{1, \dots, n\}$, then WLOG $\sigma(i) = i$. Thus, $\sigma(\tau(i)) \underset{\uparrow}{=} \tau(i) = \tau(\sigma(i))$. $\square$

(disjointness)

Note: There are many ways to describe an element of S_n. e.g. $(12)(23) = (123)$, but we will prove later that there is a unique way to express each element as a product of disjoint cycles (up to reordering cycles, and cyclically permuting cycle elements), given by the output of the above algorithm.

Optional coding exercise:

1.) Write a function that takes as input a permutation of $1, \dots, n$ (e.g. interpret 4231 as the bijection $\begin{matrix} 1 \mapsto 4 \\ 2 \mapsto 2 \\ 3 \mapsto 3 \\ 4 \mapsto 1 \end{matrix}$) and outputs the cycle decomposition.

2.) Write a function that multiplies cycles.